Turing Machine

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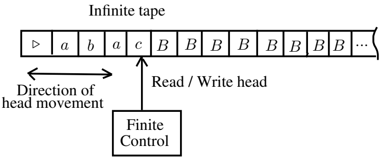
Alan M. Turing

- Alan Turing was one of the founding fathers of CS.
- His computer model - the Turing Machine (TM) - was inspiration /premonition of the electronic computer which came two decades after the TM model
- Invented the *Turing Test* for in AI
- Legacy: *The Turing Award, eminent award in CS research*

Definition

(Church-Turing Thesis:) TM is ultimate model for compuation. Any thing, which is solvable, i.e., has an algorithm, what ever the computation model is used to compute that algorithm, it is ultimately the TM model.

Turing Machine Model for computation



$$M = (Q, \Sigma, \Gamma, \delta, s, H),$$

where, Q is set of states
H is set of Halting states, $H \subseteq Q$
Σ is set of input symbols
Γ is tape alphabet, $\Gamma = \Sigma \cup \{B, \triangleright\}$
δ is transition function (a partial function),
$\delta : (Q - H) \times \Gamma \to Q \times \Gamma \times \{L, R\}$

Definition

Language acceptability by TM:
$L = \{w | w \in \Sigma^*, q_0 w \vdash^* \alpha p \beta, p \in H; \alpha, \beta \in \Gamma^*\}$

- TM has a infinite amount of read / write memory. Input is assumed to reside on the tape.
- 1936(Alan M. Turing): Any logical/arithmetic computation, for which complete instructions for carrying out this are supplied, it is always possible to design a TM that can perform this computation.
- TM v/s Human: TM model is based on human problem solving process using pencil and paper. As we do this, our mental state changes, for every smallest step. Correspondingly, TM has tape (=paper), R/W head (=pencil), and state (=state of our mind).
- $\{a^nb^n|n \geq 0\}$ v/s $\{a^nb^nc^n|n \geq 0\}$
- Powers of TM: Power is problem solving capability, and not about how fast or slow it can do.

Turing Machine

These are Automata, as simple as possible - to define formally, describe and reason about them, as general as possible (any computation can be represented by them).

Definition

(Acceptability by Turing machine:) A string w is accepted by TM M if after being put on the tape with the TM head set to the left-most position, and letting M run, M eventually enters the halting state. In this case w is an element of L(M), the language accepted by M is,

$$L(M) = \{w \mid w \in \Sigma^* \land q_0 w \vdash^* \alpha y \beta\},$$

where, y is halting configuration, and $\alpha, \beta \in \Gamma^*$

Turing Machine solves a Problem: Erase entire tape

Consider a TM $M = (Q, \Sigma, \Gamma, \delta, s, H), Q = \{q_0, q_1\}$,
$\Sigma = \{a\}, \Gamma = \{a, B, \triangleright\}, s = q_0$, B is blank character, $\triangleright$ is left end marker.

$H = \{q_1\}$

$\delta(q_0, a) = (q_0, B, R)$

$\delta(q_0, B) = (q_1, B, L)$

Let $w = aaaa$

q_0aaaaB

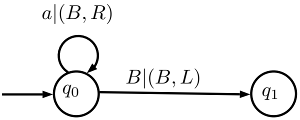
$\vdash Bq_0aaaB$

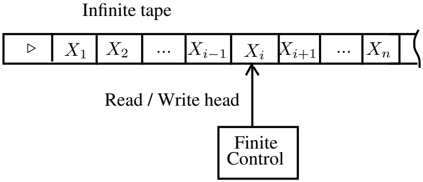
$\vdash BBq_0aaB$

$\vdash BBBq_0aB$

$\vdash BBBBq_0B$

$\vdash BBBq_1B$





Configuration-1

- A configuration of a TM: $X_1X_2\ldots X_{i-1}qX_iX_{i+1}\ldots X_n$
- Current state: (q)
- Symbols on tape: $X_1 \ldots X_n$
- position of RW head: X_i
- A formal specification of configuration:
- uqv, where u,v are strings on Σ, and uv is current content on tape, q is current state, and head is at first symbol of v.

  For example, $00101q_5011$ where read head points at *0* (third digit from end) and state is q_5.

Configuration-2

- For Two configurations:

  uaq_ibv and uq_jacv, where , $a, b, c \in \Sigma$ and $u, v \in \Sigma^*$

$$uaq_ibv \vdash uq_jacv \text{ if } \delta(q_i, b) = (q_j, c, L)$$
$$uaq_ibv \vdash uacq_jv \text{ if } \delta(q_i, b) = (q_j, c, R)$$

- Two special cases:

- The left most cell:

$$q_ibv \vdash q_jcv \text{ for } \delta(q_i, b) = (q_j, c, L)$$
$$q_ibv \vdash cq_jv \text{ for } \delta(q_i, b) = (q_j, c, R)$$

- On the cell with blank symbol:

  uaq_i is equivalent to uaq_iB

Example: of language recognition

Design TM to accept: $\{a^n b^n \mid n \geq 0\}$

Let $M = (Q, \Sigma, \Gamma, \delta, s, H)$. The algorithm can be specified as:

1. M replaces left most 'a' by 'X', and then head moves to right until it encounters left most b
2. Replaces this b by Y, and then moves left to find the right most X. Then moves one step right to left most a
3. Repeat Step 2 and 3 in order, i.e., 2, 3, 2, 3, ...
4. When searching for b, if finds a blank character B (i.e., $|a^n| > |b^n|$), then M does not accept w.
5. If a is not found but it finds b, then also M does not accept, (i.e., $|a^n| < |b^n|$).
6. After changing last b to Y, if M finds no more a then it checks that no more b remains. If this is true then $a^n b^n$ is accepted by M i.e., $|a^n| = |b^n|$

Example: of language recognition

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{\triangleright, a, b, X, Y, B\}$$
$$s = q_0$$
$$H = \{q_4, q_0\}$$

- Design TM to accept: $\{a^n b^n \mid n \geq 0\}$

1. $\delta(q_0, a) = (q_1, X, R)$
   $\delta(q_1, a) = (q_1, a, R)$; skip through $a's$ and
   $\delta(q_1, Y) = (q_1, Y, R)$; then $Y's$
   $\delta(q_1, b) = (q_2, Y, L)$
   $\delta(q_2, Y) = (q_2, Y, L)$, traverse through Y's and then
   $\delta(q_2, a) = (q_2, a, L)$, traverse a's

TM to accept: $\{a^n b^n \mid n \geq 0\}$

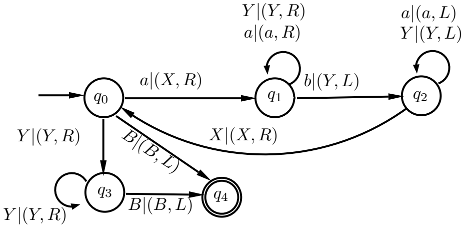


Figure 1: Transition diagram for TM accepting Language $L = \{a^n b^n \mid n \geq 0\}$.

- Move from R to L until X is found and start back:

  $\delta(q_2, X) = (q_0, X, R)$, right most X is found. Now repeat from 1 else from 2.

2. $\delta(q_0, Y) = (q_3, Y, R)$, scan through Y's

   $\delta(q_3, B) = (q_4, B, L)$, accept w, and halt.

Transition Table for $\{a^n b^n \mid n \geq 0\}$

Current State	Tape Symbol a	Tape Symbol b	Tape Symbol B	Tape Symbol X	Tape Symbol Y
q_0	(q_1, X, R)	-	(q_4, B, L)	-	(q_3, Y, R)
q_1	(q_1, a, R)	(q_2, Y, L)	-	-	(q_1, Y, R)
q_2	(q_2, a, L)	-	-	(q_0, X, R)	(q_2, Y, L)
q_3	-	-	(q_4, B, L)	-	(q_3, Y, R)
q_4	-	-	-	-	-

Example: of language recognition Dry Run

- TM to accept: $\{a^n b^n \mid n \geq 0\}$, Let $w = aabb$

$$
\begin{aligned}
q_0 aabbB &\vdash Xq_1 abbB \vdash Xaq_1 bbB \vdash Xq_2 aYbB \vdash Xq_2 aBbB \\
&\vdash q_2 XaYbB \vdash Xq_0 aYbB \vdash XXq_1 YbB \vdash XXYq_1 bB \\
&\vdash XXq_2 YYB \vdash Xq_2 XYYB \vdash XXq_0 YYB \vdash XXYq_3 YB \\
&\vdash XXYYq_3 B \vdash XXYq_4 YB \text{ (accept the input)}
\end{aligned}
$$

Total number of transitions for $|w| = n$ are: $n/2$ forward and $n/2$ in backward, in each to and fro round, i.e., n. Since, in each trip, two symbols are marked, therefore, there will be total $n/2$ trips, making total transitions: $n \times n/2 = n^2/2$. Time complexity, $O(n^2/2) = O(n^2)$, which is polynomial (P) time complexity.